Exact Solutions and Solitons with Fission and Fusion Properties for the Generalized Broer-Kaup System

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Z. Naturforsch. **61a,** 16 – 22 (2006); received December 19, 2005

Starting from a Painlevé-Bäcklund transformation and a linear variable separation approach, we obtain a quite general variable separation excitation to the generalized (2+1)-dimensional Broer-Kaup (GBK) system. Then based on the derived solution, we reveal soliton fission and fusion phenomena in the (2+1)-dimensional soliton system. – PACS numbers: 05.45.Yv, 03.65.Ge

Key words: GBK System; Exact Solution; Soliton Fission; Soliton Fusion.

1. Introduction

It is well-known that many dynamical problems in physics and other natural fields are usually characterized by nonlinear evolution partial differential equations known as governing equations. In soliton theory, searching for an analytical exact solution to a nonlinear physical system has long been an important and interesting topic both for physicists and mathematicians since much physical information and more insight into the physical aspects of a nonlinear problem can be derived from the analytical solution and thus lead to further potential applications. In the recent decades, much work has been done on this subject of looking for exact solutions and their related properties such as evolutional behavior of an interaction solution for a nonlinear physical model. For instance, a discovery via the Bäcklund transformation by Boiti et al. [1] of dromion-type localized coherent solutions of the Daver-Stewartson system provided renewed interest in higher dimensional soliton systems.

Conventionally, a collision between solitons of integrable models is regarded to be completely elastic [2]. That is to say, the amplitude, velocity and wave shape of a soliton do not undergo any change after the nonlinear interaction [3, 4]. However, for some special solutions of certain (2+1)-dimensional models in our colleagues' and our own recent study, the interactions among solitonic excitations like peakons and compactons are not completely elastic since their shapes or amplitudes are changed after their collisions [5, 6]. Furthermore, for some (1+1)-dimensional models, two or more solitons may fuse into one soliton at a special time while sometimes one soliton may fission into two or more solitons at other special times [7]. These phenomena are often called soliton fusion and soliton fission, respectively. Actually, the soliton fusion and fission phenomena have been observed in many physical systems, such as organic membrane and macromolecular material [8], and physical fields, like plasma physics, nuclear physics and hydrodynamics [9]. Recently, Zhang et al. [10] and Lin et al. [11] studied the evolutions of soliton solutions for two (1+1)-dimensional PDEs with time and revealed the soliton fission and soliton fusion phenomena. In a similar way, Wang et al. [12] further discussed two (1+1)-dimensional models, the Burgers equation and the Sharma-Tasso-Olver equation, via Hirota's direct method, and also found the soliton fission and soliton fusion phenomena. Now an important and interesting problem is if are there soliton fission and fusion phenomena in higher dimensions? The main purpose of our present paper is searching for some possible soliton fission and soliton fusion phenomena in (2+1)-dimensions. As a special example, we consider the generalized (2+1)-dimensional Broer-Kaup system (GBK) [13]

$$h_t - h_{xx} + 2hh_x + u_x + Au + Bg = 0, \tag{1}$$

$$g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy}) + 4B(g_y - h_{yy}) + C(g - 2h_y) = 0, \tag{2}$$

$$u_y - g_x = 0, \tag{3}$$

where A, B, C are arbitrary constants. The GBK system is derived from a typical (1+1)-dimensional Broer-Kaup (BK) system [14, 15]

$$u_t - uu_x - v_x + \frac{1}{2}u_{xx} = 0, \quad v_t - (uv)_x - \frac{1}{2}v_{xx} = 0, \tag{4}$$

which is usually used to describe the propagation of long waves in shallow water [16] by means of the Painlevé analysis. In [17], Huang and Zhang derived a simplified GBK system (SGBK) through differentiating (1) with respect to variable y once and substituting (3) into (1) when setting $B = 0$:

$$(h_t - h_{xx} + 2hh_x)_y + g_{xx} + Ag_x = 0, \tag{5}$$

$$g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy}) + C(g - 2h_y) = 0. \tag{6}$$

Some exact soliton-like solutions of the SGBK system are found by a variable-coefficient projective Riccati equation method [17].

Obviously, when $A = B = C = 0$, the GBK system will degenerate the (2+1)-dimensional Broer-Kaup system (BK) [18], which can be derived from inner parameter dependent symmetry constraints of the Kadomtsev-Petviashvili model [19]. Using some suitable dependent and independent variable transformations, Chen and Li [20] have proved that the (2+1)-dimensional BK system can be transformed to the (2+1)-dimensional dispersive long wave equation (DLWE) [21] and (2+1)-dimensional Ablowitz-Kaup-Newell-Segur system (AKNS) [22]. Actually, the (2+1)-dimensional BK system has been widely investigated by many researchers [23–26]. However, to the best of our knowledge, the studies on the general solution and, especially, the soliton fission and soliton fusion phenomena for the (2+1)-dimensional GBK system have not been reported in the preceding literature.

2. Exact Solutions for the (2+1)-Dimensional GBK System

In this section, we will give a quite general solution of the GBK system. As is known, to search for solitary wave solutions to a given nonlinear partial differential model, one can utilize different approaches. One of the useful and powerful methods is an extended homogeneous balance approach (EHBA) [27]. According to the EHBA principle (which can also be obtained through the standard truncated Painlevé expansion [3]), let us begin with a Painlevé-Bäcklund transformation for h, g, u in (1), (2) and (3):

$$h = (\ln f)_x + h_0, \quad g = 2(\ln f)_{xy} + g_0, \quad u = 2(\ln f)_{xx} + u_0, \tag{7}$$

where $f = f(x,y,t)$ is an arbitrary function of the variables $\{x,y,t\}$ to be determined and $\{h_0, g_0, u_0\}$ are three seed solutions satisfying the GBK system. In usual cases, by choosing some special trivial solutions, we can directly obtain the seed solutions. In the present case, we do some simple calculations. For convenience in the following discussions, we choose the seed solutions $\{h_0, g_0, u_0\}$ to be $\{h_0 = h_0(x,t),\ g_0 = 0,\ u_0 = u_0(x,t)\}$ where $h_0(x,t)$ is an arbitrary function of the indicated arguments while $u_0(x,t)$ is fixed by the following equation:

$$h_{0t} - h_{0xx} + 2h_0h_{0x} + u_{0x} + Au_0 = 0. \tag{8}$$

Based on (8), one can easily obtain:

$$u_0(x,t) = \mathrm{e}^{-Ax}[\sigma(t) - \int^x \mathrm{e}^{Ax}\chi(x,t)\mathrm{d}x], \tag{9}$$

where $\sigma(t)$ is an arbitrary function of t and $\chi(x,t) = h_{0t} - h_{0xx} + 2h_0h_{0x}$.

Substituting (7) together with the seed solutions into (1) and (2) yields

$$(f^2\partial_x - ff_x)(f_t + f_{xx} + 2Af_x + 2Bf_y + 2h_0f_x) = 0, \tag{10}$$

$$[f^2\partial_{xy} - f(f_x\partial_y + f_y\partial_x + f_{xy}) + 2f_xf_y] \cdot (f_t + f_{xx} + 2Af_x + 2Bf_y + 2h_0f_x) = 0, \tag{11}$$

while (3) is satisfied identically under the above Painlevé-Bäcklund transformation (7).

It can be easily seen that if f satisfies

$$f_t + f_{xx} + 2Af_x + 2Bf_y + 2h_0f_x = 0, \tag{12}$$

then (10) and (11) are satisfied automatically. In [13], Zhang et al. take f as such an ansatz

$$f = f_1(x,t) + f_2(y,t), \tag{13}$$

and derive some special solutions for the GBK system. In usual cases, one can suppose that f has the following variable separated form [3, 4]:

$$f = a_0 + a_1p(x,t) + a_2q(y,t) + a_3p(x,t)q(y,t). \tag{14}$$

In our present paper, we try to obtain a more general solution for the GBK system by choosing a more general ansatz for f. Since (12) is a linear equation, one can certainly use the linear superposition theorem. For instance

$$f=\lambda+\sum_{k=1}^{N}P_k(x,t)Q_k(y,t), \tag{15}$$

where λ is an arbitrary constant, $P_k(x,t)\equiv P_k$ and $Q_k(y,t)\equiv Q_k$ $(k=1,2,\cdots,N)$ are variable separated functions of $\{x,t\}$ and $\{y,t\}$, respectively. It is evident that the ansatz (13) or (14) is a special case of the general ansatz (15).

Inserting the ansatz (15) into (12) yields the following set of variable separated equations:

$$P_{kt}+2h_0P_{kx}+P_{kxx}+2AP_{kx}+\Gamma_k(t)P_k=0, \tag{16}$$

$$Q_{kt}+2BQ_{ky}-\Gamma_k(t)Q_k=0, \tag{17}$$

where $\Gamma_k(t)$ $(k=1,2,\cdots,N)$ are arbitrary functions of time t. Then a general variable separation excitation for the GBK system yields

$$h=\frac{\sum_{k=1}^{N}P_{kx}Q_k}{\lambda+\sum_{k=1}^{N}P_kQ_k}+h_0, \tag{18}$$

$$g=\frac{2\sum_{k=1}^{N}P_{kx}Q_{ky}}{\lambda+\sum_{k=1}^{N}P_kQ_k}-\frac{2\sum_{k=1}^{N}P_{kx}Q_k\sum_{k=1}^{N}P_kQ_{ky}}{(\lambda+\sum_{k=1}^{N}P_kQ_k)^2}, \tag{19}$$

$$u=\frac{2\sum_{k=1}^{N}P_{kxx}Q_k}{\lambda+\sum_{k=1}^{N}P_kQ_k}-\frac{2(\sum_{k=1}^{N}P_{kx}Q_k)^2}{(\lambda+\sum_{k=1}^{N}P_kQ_k)^2}+u_0. \tag{20}$$

Here h_0, P_k and Q_k solve (16) and (17), and u_0 is expressed by (9).

In order to discuss some interesting properties of the above general excitations (18), (19) and (20), we make further simplifications and give some special exact solutions.

Case 1. We first consider the simplest case: $N=1$, $\{P_1,Q_1\}=\{P,Q\}$, $\Gamma_1(t)=\tau(t)$. Then (15), (16) and (17) become:

$$f=\lambda+PQ, \tag{21}$$

$$P_t+P_{xx}+2h_0P_x+2AP_x+\tau(t)P=0, \tag{22}$$

$$Q_t+2BQ_y-\tau(t)Q=0. \tag{23}$$

It is easy to obtain the solution of (22). Since $h_0(x,t)$ is an arbitrary seed solution, we can consider P as an arbitrary function of $\{x,t\}$, and then the seed solution h_0 is determined by (22):

$$h_0=-\frac{P_t+2AP_x+P_{xx}+\tau(t)P}{2P_x}. \tag{24}$$

As to (23), its general solution has the form

$$Q(y,t)=S(y-2Bt)\exp\int^t\tau(t)\mathrm{d}t, \tag{25}$$

where $S(y-2Bt)\equiv S$ is an arbitrary function of $\{y-2Bt\}$.

Finally, we can derive a special variable separated excitation of the (2+1)-dimensional GBK system:

$$h=\frac{P_xS\exp\int^t\tau(t)\mathrm{d}t}{\lambda+PS\exp\int^t\tau(t)\mathrm{d}t}+h_0, \tag{26}$$

$$g=\frac{2\lambda P_xS_y\exp\int^t\tau(t)\mathrm{d}t}{[\lambda+PS\exp\int^t\tau(t)\mathrm{d}t]^2}, \tag{27}$$

$$u=\frac{2P_{xx}S\exp\int^t\tau(t)\mathrm{d}t}{\lambda+PS\exp\int^t\tau(t)\mathrm{d}t}-2\left(\frac{P_xS\exp\int^t\tau(t)\mathrm{d}t}{\lambda+PS\exp\int^t\tau(t)\mathrm{d}t}\right)^2+u_0, \tag{28}$$

with three arbitrary functions $P(x,t)$, $S(y-2Bt)$ and $\tau(t)$, and h_0, u_0 are expressed by (24) and (9), respectively.

Case 2. In a similar way, when we consider the case: $N=3$, $\lambda=a_0$, $\{P_1,Q_1\}=\{p(x,t),a_1\}$, $\{P_2,Q_2\}=\{a_2,q(y,t)\}$, $\{P_3,Q_3\}=\{p(x,t),a_3q(y,t)\}$, $\Gamma_k(t)=0$, here a_i $(i=1\ldots4)$ are arbitrary constants, then (15), (16) and (17) become:

$$f=a_0+a_1p+a_2q+a_3pq, \tag{29}$$

$$p_t+p_{xx}+2h_0p_x+2Ap_x=0, \tag{30}$$

$$q_t+2Bq_y=0. \tag{31}$$

Based on (29), (30) and (31), one can obtain another special exact excitation:

$$h=\frac{p_x(a_1+a_3q)}{a_0+a_1p+a_2q+a_3pq}+h_0, \tag{32}$$

$$g=\frac{2(a_3a_0-a_2a_1)p_xq_y}{(a_0+a_1p+a_2q+a_3pq)^2}, \tag{33}$$

$$u=\frac{2(a_1+a_3q)p_{xx}}{a_0+a_1p+a_2q+a_3pq}-\frac{2(a_1+a_3q)^2p_x^2}{(a_0+a_1p+a_2q+a_3pq)^2}+u_0, \tag{34}$$

with two arbitrary functions $p(x,t) \equiv p$, $q(y-2Bt) \equiv q$, and u_0 is expressed by (9) while $h_0 = -\frac{p_t+2Ap_x+p_{xx}}{2p_x}$. It is interesting to mention that the previously derived result in [10] is a special case of this case 2 when setting $a_3 = a_0 = 0$ and $a_2 = a_1 = 1$.

3. Fission and Fusion of Localized Excitations in the GBK System

Now comparing the special solution g expressed by (33) with the so-called common formula (1.1) in [3], one might be surprised to find that they are identical. Therefore, all the localized excitations based on the common formula (1.1), such as dromions, lumps, brethers, instantons, fractal and chaotic patterns obtained in[3], can be re-derived in the GBK system. Since these localized structures have been widely reported in the previous literature [3, 28–30], we neglect the related discussions in this section.

We do not study the general field g expressed by (19), but only discuss the simplest excitation g expressed by (27). Actually, even in this simplest situation, one can still find rich localized structures for the (2+1)-dimensional GBK system. As the arbitrariness of characteristic functions P, S and $\tau(t)$ is included in the special field (27), g may possess quite rich structures such as peakons, compactons and foldons when we select the arbitrary functions P, S and $\tau(t)$ appropriately to avoid singularities, which also implies that some exotic behaviors can propagate along the above lines.

From above brief discussions, one may conclude that the field g comprises some novel properties that have not been revealed until now. Recently, it has been reported both theoretically and experimentally that fission and fusion phenomena can happen for (1+1)-dimensional solitons or solitary waves [12]. Now we focus our attention on these intriguing fusion and fission phenomena for the special field g in (2+1)-dimensions, which may exist in certain situations. For instance, if we select the arbitrary functions P and S to be

$$P(x,t) = 1+2\exp(x-2t) + \begin{cases} \exp(x+t), \\ \quad \text{if } (x+t) \le 0, \\ -\exp(-x-t)+2, \\ \quad \text{if } (x+t) > 0, \end{cases} \quad (35)$$

$$S(y-2Bt) = 1+\exp(y-2Bt), \quad B = 1, \quad (36)$$

and $\lambda = 1$, $\tau(t) = 0$ in (27), then we can obtain a new kind of fissioning solitary wave solution for the field g (27). In Fig. 1 a possible profile of the solitary wave solution for the field g is shown and a fission phenomenon is depicted. From Figs. 1a, 1b, 1c and 1d, one can clearly see that one soliton fissions into two solitons. It is interesting to mention that the travelling soliton in Fig. 1e, i.e. one of the pairs of solitons that emerge after the fission, is stable and do not undergo additional fission at least not if running the program for longer periods of time up to $t = 10^3$. However, the travelling soliton in Fig. 1f is unstable and will fission further into many oscillating solitons as time $t > 12$, their shapes and amplitudes are changing with time.

Along the above line, when we consider P and S to be

$$P(x,t) = 1+\left[\exp(5x-5t)+0.9\exp(2x-3t) + 1.2\exp(2x-4t)\right] \cdot \left[1+\exp(2x-3t)\right]^{-2}, \quad (37)$$

$$S(y-2Bt) = 1+\exp(y-2Bt), \; B = 1, \quad (38)$$

and $\lambda = 1$, $\tau(t) = 0$ for the special field g (27), we can obtain another new type of fusion solitary wave, which possesses apparently different properties as compared to Figure 1. From Fig. 2, one can find that three solitons fuse into one soliton finally. The fused single soliton remains stable for subsequent times if the program runs for rather long time ($t = 10^3$).

Certainly, if we consider other choices for the arbitrary functions P, S and $\tau(t)$ such as Jacobian functions or the solutions of the well-known Lorenz chaotic system, then we may derive some novel solitary wave solutions with double periodic properties or chaotic behaviors, which are omitted in our present paper. Actually, because there exist some arbitrary characteristic functions P, S and $\tau(t)$ in the special field g, any exotic behavior may engender along with the above mentioned lines.

4. Summary and Discussion

In summary, with the aid of an extended homogeneous balance approach and a linear variable separation method, the (2+1)-dimensional GBK system is successfully solved. Based on a special solution of the derived general solution with arbitrary functions, we list two simple examples, soliton fission and fusion solutions for the (2+1)-dimensional GBK system. Conventionally, it is considered that the interactions among

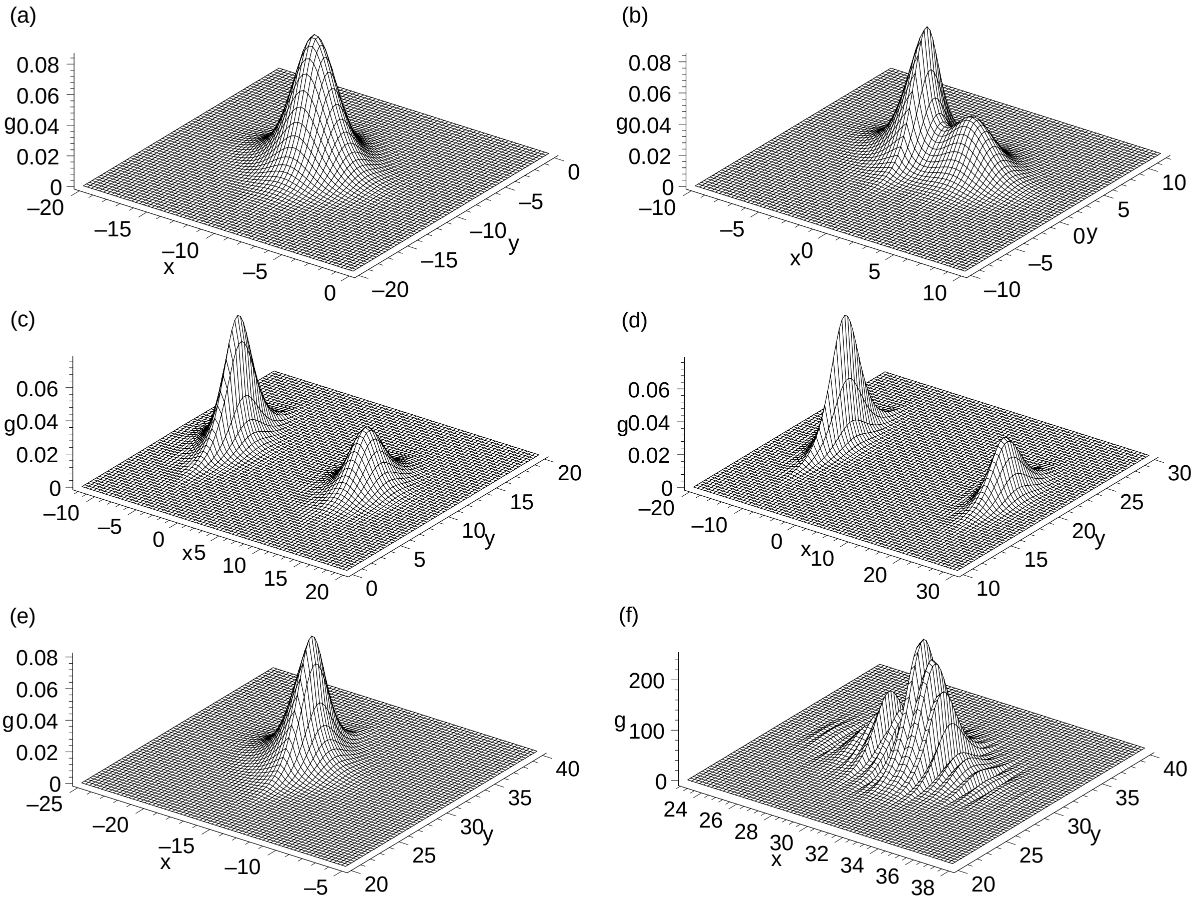

Fig. 1. The time evolution of one soliton fission into two solitons for the field g (27) with the conditions (35) and (36) at different times: (a) $t = -5$; (b) $t = 1$; (c) $t = 5$; (d) $t = 10$. (e) The stable left travelling soliton at $t = 15$. (f) The unstable right travelling soliton at $t = 15$.

solitons are completely elastic. However, in some special cases, the soliton collisions may be nonelastic or completely nonelastic. From the brief analysis in our present paper, we can see that these intriguing phenomena like the soliton fission and fusion can occur in a higher dimensional soliton system if we choose appropriate initial conditions or boundary conditions, which are similar to some work in (1+1)-dimensions carried out by several authors [10 – 12]. Although we have reported about some soliton fusion and fission phenomena in (2+1)-dimensions, it is obvious that there are still many significant and interesting problems waiting for further discussions. One may be puzzled about how the solution looses its property to have several peaks and continues with only one peak and vice versa? How do these phenomena depend on the particular choice of the parameters? Just as the authors [10 – 12] have pointed out in the (1+1)-dimensional case, what are the necessary and sufficient conditions for soliton fission and soliton fusion? What is the general equation for the distribution of the energy and momentum after soliton fission and soliton fusion? How can we use the soliton fission and soliton fusion of integrable models to investigate practically observed soliton fission and soliton fusion in the experiments? These are all the pending issues. Actually, our present short paper is merely an initial work, due to a broad variety of potential applications of soliton theory. To learn more about the soliton fission and fusion properties and their applications in reality is worthy to be studied further.

Acknowledgements

The authors would like to thank the referees for valuable suggestions. C.-L. Z. is indebted to the help-

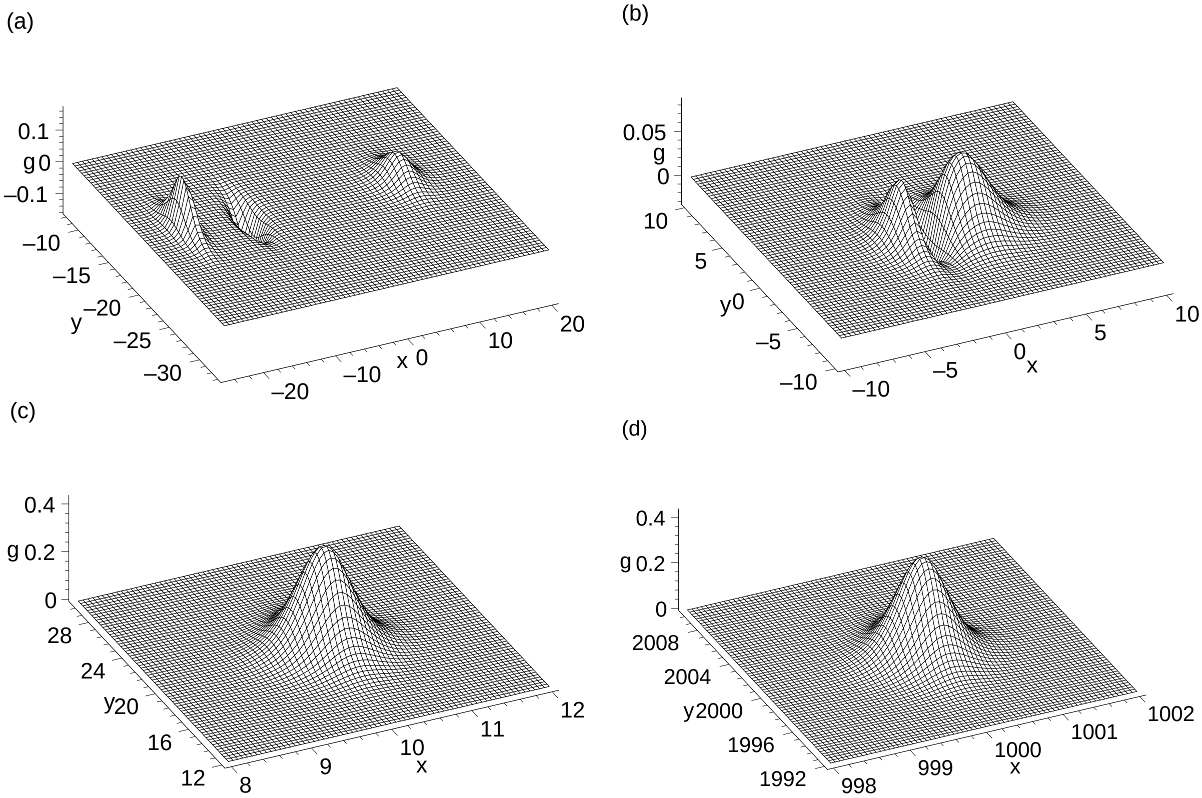

Fig. 2. Three solitons fuse into one soliton: time evolution of the field g (27) with conditions (37) and (38) at different times: (a) $t = -10$; (b) $t = -1$; (c) $t = 10$; (d) $t = 1000$.

ful discussions with Profs. S. Y. Lou and J. F. Zhang. This work was supported by Natural Science Foundation of Zhejiang Province (Grant No. Y604106), the Foundation of New Century “151 Talent Engineering” of Zhejiang Province, the Scientific Research Foundation of Key Discipline of Zhejiang Province, and the Natural Science Foundation of Zhejiang Lishui University.

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